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# Uniqueness in Restricted Range Approximation with Betweenness

CHARLES B. DUNHAM

*Computer Science Department, University of Western Ontario, London, Ontario N6A 5B9, Canada*

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Let $X$ be a compact Hausdorff space. Let $\mathscr{G}$ be a subset of $C(X)$ with the betweenness property [2]. Let $u$ and $v$ be continuous into the extended real line, $u < v$. We consider uniqueness of best Chebyshev approximations under the constraint on approximations $G$ that

$$u \leqslant G \leqslant v.$$

This approximation problem was first considered in [3].

DEFINITION. $\mathscr{G}$ has *zero-sign compatibility* if, for any two distinct elements $G$, $H$, any closed set $Z$ of zeros of $G - H$, and any continuous function $s$ taking the values $+1$ or $-1$ on $Z$, there exists $F \in \mathscr{G}$ such that

$$\operatorname{sgn}(F(x) - G(x)) = s(x), \qquad x \in Z.$$

Zero-sign compatibility is necessary for uniqueness in ordinary best Chebyshev approximation and sufficient for uniqueness in ordinary best Chebyshev approximation if $\mathscr{G}$ has the betweenness property [2].

DEFINITION. A normal topological space in which each closed set is a countable intersection of open sets is called perfectly normal.

The perfectly normal spaces include all subsets of finite dimensional Euclidean space.

THEOREM. *Let $X$ be perfectly normal. If $\mathscr{G}$ has zero-sign compatibility, best restricted range approximations are unique for $f$ in $[u, v]$.*

*Proof.* Suppose $H$ and $I$ are distinct best approximations to $f$. By



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arguments of Lemmas 3, 5 of [2], by taking $G$ in the $\lambda$-set of $H$, $I$ for any $\lambda$ in $(0, 1)$, we get $G$ also best and $G$, $H$ agree on $\hat{M}(G)$, where

$$\hat{M}(G) = \{x: |f(x) - G(x)| = \|f - G\|\} \bigcup \{x: u(x) = G(x)\}$$

$$\bigcup \{x: v(x) = G(x)\}.$$

Define

$$\begin{aligned} s(x) &= \operatorname{sgn}(f(x) - G(x)) && |f(x) - G(x)| = \|f - G\| \\ &= +1 && G(x) = u(x) \\ &= -1 && G(x) = v(x). \end{aligned}$$

No inconsistency can arise in the above definition. Suppose, for example, we had $f(x) - G(x) = -\|f - G\| < 0$ and $G(x) = u(x)$, then $f(x) < u(x)$ and $f \notin [u, v]$.

Now $x$ such that $s(x) = -1$ and $x$ such that $s(x) = +1$ form disjoint closed sets. By a result of Dugundji [1, p. 148], there is a continuous extension of $s$ to $X$ such that $|s(x)| < 1$ for all other $x$. By the definition of zero-sign compatibility, there is $F \in \mathscr{G}$ with

$$\operatorname{sgn}(F(x) - G(x)) = s(x) \qquad x \in \hat{M}(G).$$

But this contradicts Theorem 2 of [3].

COROLLARY. *Let $X$ be perfectly normal. If best approximations are unique in ordinary Chebyshev approximation, they are unique in the restricted range problem if $f \in [u, v]$.*

In the case of one-sided approximation from above, with $u = f$, $v = +\infty$, an explicit extension for $s$ is available and we do not need to assume perfect normality. Choose

$$s(x) = 1 - 2\,|f(x) - G(x)|/\|f - G\|$$

then $-1 \leqslant s \leqslant 1$ with equality only if $G(x) - f(x) = \|f - G\|$ or $G(x) = f(x)$. One-sided approximation from below is handled similarly.

## References

1. J. Dugundji, "Topology," Allyn & Bacon, Boston, 1968.
2. C. B. Dunham, Chebyshev approximation by families with the betweenness property, *Trans. Amer. Math. Soc.* **136** (1969), 151–157.
3. C. B. Dunham, Chebyshev approximation with restricted range by families with the betweenness property, *J. Approx. Theory* **11** (1974), 254–259.